ISRAEL JOURNAL OF MATHEMATICS, Vol. 29, Nos. 2-3, 1978

INCOMPLETE POLYNOMIALS OF BEST APPROXIMATION†

BY

G. G. LORENTZ

ABSTRACT

The main theorem proved in this paper is as follows. There exist odd functions $f \in C[-1,1]$ with the following property. Let P_n be the polynomial of best uniform approximation to f of degree $\leqq n$. Then for infinitely many n, P_n has zero of order $s(n) \geqq c \log n$ at $x = 0$.

§1. Lemmas and intermediate results

We use the term "incomplete polynomial" to denote a polynomial of the form

$$P_n(x) = \sum_{k=s}^{n} a_k x^k, \tag{1.1}$$

where $s > 0$. We would like to have s as large as possible. In [1] we have proved, among other things, the existence of a function $0 < \Delta(\theta) < 1$, defined for $0 < \theta < 1$ with the following property. If $P_n(x)$ is a sequence of polynomials (defined for infinitely many n) of form (1.1) with $s = s(n) \geqq \theta n$, and if $|P_n(x)| \leqq 1$ on $[0,1]$, then $P_n(x) \to 0$ uniformly on each interval $[0,\delta]$, $\delta < \Delta(\theta)$; this is not always true for $\delta > \Delta(\theta)$. The exact value of $\Delta(\theta)$ is not known, but we have shown in [1] that $\theta^2 \leqq \Delta(\theta) < \theta$, $0 < \theta < 1$. Moreover, R. Varga (private communication) finds that

$$\overline{\lim_{\theta \to 0}} \frac{\Delta(\theta)}{\theta^2} \leqq \frac{3\pi^2}{16} = 1.850\ldots.$$

In the present paper, we investigate the possibility that a given function $f \in C[-1,+1]$ has incomplete polynomials as its polynomials of best approximation. Our main result is that $s(n) \geqq \text{const} \cdot \log n$ can happen infinitely often for such polynomials.

† This research has been supported by Grant MPS 75-09833 of the National Science Foundation.
Received May 3, 1977 and in revised form June 30, 1977

The problem solved here has been formulated jointly by J. Blatter and the author at IMPA (Rio de Janeiro) in the Summer of 1976. I am indebted to K. Zeller for the idea of using series (3.1) in this problem.

THEOREM 1. *Let P_n, P_{n+1} be polynomials of best approximation (of degrees $\leqq n$ and $\leqq n+1$, respectively) to $f \in C[a,b]$, and let $P_n \neq P_{n+1}$. Then $Q = P_{n+1} - P_n$ has $n+1$ distinct real zeros, which lie in the open interval (a,b).*

PROOF. Let $a \leqq x_1 < x_2 < \cdots < x_{n+2} \leqq b$ be $n+2$ points of the Čebyšev alternance for P_n; if for instance $f(x_i) - P_n(x_i) > 0$, then

$$f(x_i) - P_n(x_i) = \|f - P_n\| > \|f - P_{n+1}\| \geqq f(x_i) - P_{n+1}(x_i),$$

so that $Q(x_i) < 0$. Similarly, $Q(x_{i+1}) > 0$. Thus, Q changes sign on each of the intervals $[x_i, x_{i+1}]$.

REMARK. The same proof yields, for two polynomials $P_n \neq P_m$, $m > n$ of best approximation, the fact that P_m and P_n cannot have a common root of multiplicity $> m - n$.

To S. Bernstein we owe the observation that if $b_k \geqq 0$, $\Sigma b_k < +\infty$, then all partial sums of $f = \Sigma_0^\infty b_k T_{3^k}$ (where T_n denotes the n-th Čebyšev polynomial) are polynomials of best approximation to f on $[-1, 1]$. It is important to know that the restriction $b_k \geqq 0$ is not essential.

THEOREM 2. *Let $n_j, p_j, j = 0, 1, \cdots$ be odd positive integers so that n_{j-1} divides n_j.*

Let the b_j satisfy $\Sigma_0^\infty |b_j| < +\infty$, and let

$$f(x) = \sum_{j=0}^{\infty} b_j T_{n_j}(x)^{p_j}. \tag{1.2}$$

If for an integer $k \geqq 1$

$$n_{k-1} p_{k-1} + 2 \leqq n_k, \tag{1.3}$$

then the sum $P = \Sigma_0^{k-1} b_j T_{n_j}^{p_j}$ is the polynomial of best approximation of f among all polynomials of degree $\leqq n_k - 2$ (hence also among all polynomials of degree $\leqq n_{k-1} p_{k-1}$).

PROOF. Let $g(x) = \Sigma_{j=k}^\infty b_j T_{n_j}^{p_j}$; we consider the function h on the circle T given by

$$h(t) = g(\cos t) = \sum_{j=k}^{\infty} b_j \cos^{p_j} n_j t, \quad t \in T.$$

We may assume that g and h are not identically zero. Noticing that $\frac{1}{2}(n_j/n_k+1)$ is an integer for $j \geqq k$, we have, with $c = \pi/(2n_k)$,

$$h(c \pm t) = \sum_{j=k}^{\infty} b_j \varepsilon_j \sin^{p_j}(\pm n_j t), \qquad \varepsilon_j = (-1)^{\frac{1}{2}(n_j/n_k+1)p_j}.$$

Hence $h(c-t) = -h(c+t)$, that is, h is odd at the point $c \in T$. It follows that h has a maximum $M > 0$ and a minimum $-M$ on T.

Now h has period $2\pi/n_k$, hence M and $-M$ are taken on each of the n_k intervals

$$\left[\frac{2\pi l}{n_k}, \frac{2\pi(l+1)}{n_k}\right) \quad l = 0, \cdots, n_k - 1.$$

There must be a monotone sequence of $2n_k$ points on T, where h takes alternatively the values $+M$ and $-M$. On at least one of the intervals $[0,\pi]$, $[-\pi,0]$, we have a sequence of this type consisting of n_k points. Since h is even, this happens on each of them. The same is true for $g(x)$, $-1 \leqq x \leqq +1$. In view of (1.3), our statement follows from the theorem of Čebyšev.

Let $f \in C[a,b]$. Is it possible that polynomials P_n of best approximation to f are incomplete polynomials (1.1) with large $s = s(n)$? This cannot happen for *all* large n. Indeed, by Theorem 1, unless P_n and P_{n+1} are identical, they cannot have 0 as a common zero of multiplicity $s \geqq 2$, and if $0 \notin (a,b)$, this cannot happen even with $s \geqq 1$. However, the phenomenon in question can happen *infinitely often*.

In §3 we prove that one can have $s = s(n) \geqq c \log n$, for infinitely many n. In the opposite direction, we have, if all integral values are taken by $s(n)$,

$$s(n) \leqq \text{const} \sqrt{n}, \quad n = 1, 2, \cdots. \tag{1.4}$$

Indeed, let n_s be selected so that $s(n) = s$ for $n = n_s$. The remark to Theorem 1 gives $n_s - n_{s-1} \geqq s - 1$, hence $n_s \geqq \frac{1}{2}s(s-1)$.

Polynomials P_n of best approximation of $f \in C[-1,+1]$ can all vanish for $x = 0$; this happens for all odd functions f. We offer the conjecture that this is the only possible case.

CONJECTURE. If all polynomials of best approximation of $f \in C[-1,+1]$ vanish at the origin, then f is odd.

A related conjecture has been formulated by I. Borosh: A function $f \in C[-1,+1]$ is odd if its polynomials of best approximation satisfy $P_{2k-1} = P_{2k}$, $k = 1, 2, \cdots$; it is even if $P_{2k} = P_{2k+1}$, $k = 0, 1, \cdots$.

§2. Estimation of Vandermonde determinants

We shall need results about the Vandermonde determinants

$$V = V(x_1,\cdots,x_n) = \det|x_k^{i-1}|,$$

with rows numbered $i = 1,\cdots,n$ and columns $k = 1,\cdots,n$. We denote by $V^{(t)} = V_{i_1,\cdots,i_t;k_1,\cdots,k_t}$ the subdeterminant of V obtained from V by removing t rows numbered $i_1 < \cdots < i_t$ and t columns $k_1 < \cdots < k_t$. We define $V^{(n)} = 1$.

LEMMA 1. *Let x_k satisfy $|x_k - x_l| \geqq 1$ for $k \neq l$. Then for $V^{(t)} = V_{i_1,\cdots,i_t;k_1,\cdots,k_t}$ one has*

$$\left|\frac{V^{(t)}}{V}\right| \leqq \frac{\prod_{j\neq k_1,\cdots,k_t}(1+|x_j|)^t}{\prod_{\substack{j\neq k_1,\cdots,k_t\\ s=1,\cdots,t}}|x_j - x_{k_s}|}, \quad t = 1,\cdots,n. \tag{2.1}$$

PROOF (by induction in n). Obviously, (2.1) is true for $t = n$.

Case 1. Let $i_1 = 1$. Then $V^{(t)} = \prod_{j\neq k_1,\cdots,k_t} x_j V_{i_2-1,\cdots,i_t-1,n;k_1,\cdots,k_t}$. The last determinant is equal to

$$\tilde{V}^{(t-1)} = \tilde{V}_{i_2-1,\cdots,i_t-1;k_2,\cdots,k_t}$$

where $\tilde{V}$ is the Vandermonde determinant of order $n-1$ formed by numbers x_j, $j \neq k_1$, so that $\tilde{V} = V_{n,k_1}$. Using product representations of V and $\tilde{V}$,

$$|\tilde{V}/V| = \prod_{j\neq k_1}|x_j - x_{k_1}|^{-1}. \tag{2.2}$$

Hence

$$\left|\frac{V^{(t)}}{V}\right| = \left|\frac{\tilde{V}^{(t-1)}}{\tilde{V}}\right| \prod_{j\neq k_1,\cdots,k_t}|x_j| \prod_{j\neq k_1}|x_j - x_{k_1}|^{-1}.$$

The product of the two last factors does not exceed $\prod_{j\neq k_1,\cdots,k_t}(1+|x_j|)/|x_j - x_{k_1}|$; the first factor we estimate by means of the inductive assumption. This gives (2.1).

Case 2. $i_t = n$. Here

$$V^{(t)} = \tilde{V}^{(t-1)} = \tilde{V}_{i_1,\cdots,k_{t-1};k_2,\cdots,k_t}, \qquad \tilde{V} = V_{n,k_1}$$

and an estimate similar to Case 1 gives again (2.1).

Case 3. $1 < i_1 \leqq i_t < n$. We can assume that there is a j, $1 \leqq j \leqq n$ distinct from all k_s. Without loss of generality let $j = 1$. We subtract column 1 of $V^{(t)}$

from each other column, reducing $V^{(t)}$ to a determinant of order $n-t-1$. In this determinant, we subtract from each row except the first the preceding row multiplied by a proper power of x_1. Let R be one of those rows, and R' the preceding row:

$$R: x_j^l - x_1^l; \quad R': x_j^{l'} - x_1^{l'}, \quad j \neq 1, k_1, \cdots, k_t.$$

Thus $l - l' = 1 + \sigma$, $\sigma \geqq 0$. The inequality $\sigma \geqq 1$ means that there are some numbers $i_s - 1$ in the gap between l' and l. Indeed, σ is precisely the number of s that satisfy $l' < i_s - 1 < l$.

The row R' is multiplied by $x_1^{\sigma+1}$ and subtracted from R. After factoring out $x_j - x_1$ from the columns, R will be replaced by the new row

$$x_j^{l-1} + x_1 x_j^{l-2} + \cdots + x_1^{\sigma} x_j^{l'}, \quad j \neq 1, k_1, \cdots, k_t. \tag{2.3}$$

Therefore

$$V^{(t)} = \prod_{j \neq 1, k_1, \cdots, k_t} (x_j - x_1) V', \qquad V' = D_0 + x_1 D_1 + \cdots + x_1^{\sigma} D_{\sigma},$$

where the determinants $D_0, D_1, \cdots, D_\sigma$ are obtained from V' by replacing row (2.3) by rows $x_j^{l-1}, x_j^{l-2}, \cdots, x_j^{l'}$ respectively. The same operation may be performed upon the D_i with respect to other rows of type (2.3) with $\sigma \geqq 1$. If their number is q, and $\sigma_1, \cdots, \sigma_q$ are the lengths of the gaps, we obtain

$$V' = \sum_{i_1=0}^{\sigma_1} x_1^{i_1} \sum_{i_2=0}^{\sigma_2} x_1^{i_2} \cdots \sum_{i_q=0}^{\sigma_q} \tilde{V}^{(t)} x_1^{i_q},$$

where $\tilde{V}^{(t)}$ are subdeterminants (with properly chosen rows and with columns $k_1, \cdots, k_t$ omitted) of the determinant $\tilde{V} = V_{n,1}$. Let $\tilde{M}(t) = \max |\tilde{V}^{(t)}|$, then

$$\begin{aligned} |V'| &\leqq \tilde{M}(t)(1 + |x_1| + \cdots + |x_1|^{\sigma_1}) \cdots (1 + |x_1| + \cdots + |x_1|^{\sigma_q}) \\ &\leqq \tilde{M}(t)(1 + |x_1|)^{\sigma_1 + \cdots + \sigma_q} \\ &= \tilde{M}(t)(1 + |x_1|)^t. \end{aligned}$$

Therefore, using again formula (2.2) with k_1 replaced by 1,

$$\begin{aligned} \left| \frac{V^{(t)}}{V} \right| &\leqq \frac{\tilde{M}(t)}{|V|} (1 + |x_1|)^t \prod_{j \neq 1, k_1, \cdots, k_t} |x_j - x_1| \\ &= \frac{\tilde{M}(t)}{|\tilde{V}|} (1 + |x_1|)^t \prod_{s=1,\cdots,t} |x_1 - x_{k_s}|^{-1}. \end{aligned}$$

To the quotient $\tilde{M}/\tilde{V}$ we can apply (2.1), and obtain the desired relation for V.

LEMMA 2. *Let* $x_k = 3^{2k+1}$, $k = 1,\cdots,n$. *Then, for some absolute constant* $M > 0$,

$$\left|\frac{V^{(t)}}{V}\right| \leqq M^t, \quad t = 0,\cdots,n. \tag{2.4}$$

PROOF. The right-hand side of (2.1) is the product of t quotiens

$$Q_s = \prod_{j\neq k_1,\cdots,k_t} (1+x_j) \Big/ \prod_{j\neq k_1,\cdots,k_t} |x_j - x_{k_s}|, \quad s = 1,\cdots,t.$$

To estimate one of them, we denote the x_j appearing in the products by $y_1,\cdots,y_q$, $q = n - t$, we put $y = x_{k_s}$, and assume that $y_1 < \cdots < y_i < y < y_{i+1} < \cdots < y_q$. Then

$$Q_s = \frac{(1+y_1)\cdots(1+y_q)}{(y-y_1)\cdots(y-y_i)(y_{i+1}-y)\cdots(y_q-y)}$$

$$= \frac{(1+y_1^{-1})\cdots(1+y_q^{-1})}{(y/y_1-1)\cdots(y/y_i-1)(1-y/y_{i+1})\cdots(1-y/y_q)}.$$

We can omit the first i factors in the denominator, as they are $\geqq 1$, and obtain

$$Q_s \leqq \prod_1^\infty \left(1+\frac{1}{3^k}\right) \Big/ \prod_1^\infty \left(1-\frac{1}{3^k}\right) = M.$$

§3. The main theorem

THEOREM 3. *There exists a function* $f \in C[-1,+1]$ *and a constant* $c > 0$ *with the property that for infinitely many* n, *the polynomial* P_n *of best approximation to* f *of degree* $\leqq n$ *has form* (1.1) *with* $s(n) \geqq c \log n$.

PROOF. The odd function f will be given by

$$f(x) = \sum_{k=1}^\infty b_k T_{3^k}(x), \quad |b_k| \leqq \frac{1}{k^2}. \tag{3.1}$$

For the Čebyšev polynomials T_{3^k} we have

$$T_{3^k}(x) = \sum_{i=0}^K c_{ik} x^{2i+1}, \quad K = \tfrac{1}{2}(3^k - 1), \quad k = 0,1,\cdots,$$

(3.2)

$$c_{ik} = (-1)^{K-i} \frac{2K+1}{K+i+1} \binom{K+i+1}{2i+1} 2^{2i}, \quad i = 0,\cdots,K$$

(see [2, p. 32]). Hence

$$c_{ik} = (-1)^{K-i}\frac{1}{(2i+1)!}2^{2i}3^k(K+i)\cdots(K-i+1)$$

$$= (-1)^{K-i}\frac{1}{(2i+1)!}2^{2i}3^{(2i+1)k}\left(1+\frac{i}{K}\right)\cdots\left(1-\frac{i-1}{K}\right).$$

We shall assume that $i \leqq k$. Then we obtain

$$c_{ik} = (-1)^{K-i}\frac{1}{(2i+1)!}2^{2i}3^{(2i+1)k}\left(1+\frac{\alpha_{ik}k^2}{3^k}\right), \quad |\alpha_{ik}| \leqq \gamma,$$

where γ is a constant. Let

$$(3.3) \qquad \begin{cases} c_{ik} = (-1)^{K-i}\dfrac{1}{(2i+1)!}2^{2i}3^{(2i+1)p}d_{ik}, \\ d_{ik} = \left(1+\dfrac{\alpha_{ik}k^2}{3^k}\right)3^{(2i+1)(k-p)}. \end{cases}$$

We will consider the determinants C, D and the Vandermonde determinant V, formed by the elements c_{ik}, d_{ik} and $3^{(2i+1)(k-p)}$ for $i = 0,\cdots,p$; $k = p,\cdots,2p$. Let $C^{(t)}$, $D^{(t)}$, $V^{(t)}$, $t = 0,1,\cdots,n$, be their subdeterminants. We first prove

$$(3.4) \qquad D \neq 0, \quad |D_{ik}/D| \leqq B,$$

where B is a constant.

We treat D as a function of $N = (p+1)^2$ variables α_{ik}, which we also denote by β_j, $j = 1,\cdots,N$. We have $|\beta_j| \leqq \gamma p^2 3^{-p}$. A partial derivative $D^{(t)}$ of D of order t with respect to some of the β_j has as its value the corresponding $V^{(t)}$, if all β_j are zero. The Taylor formula for D is therefore (with proper $V^{(t)}$):

$$D = V + \sum_{t=1}^{p+1}\sum_{|l_1+\cdots+l_N=t|}\frac{1}{l_1!}\cdots\frac{1}{l_N!}V^{(t)}\beta_1^{l_1}\cdots\beta_N^{l_N},$$

$$\left|\frac{D}{V}-1\right| \leqq \sum_{t=1}^{p+1}\sum_{l_1+\cdots l_N=t}\frac{1}{l_1!}\cdots\frac{1}{l_N!}(C\gamma p^2 3^{-p})^t$$

$$\leqq \sum_{t=1}^{p+1}\sum_{l_1+\cdots+l_N=t}\frac{(p+1)!}{l_1!\cdots l_N!(p+1-t)!}(C\gamma p^2 3^{-p})^t$$

$$= [1+(p+1)^2C\gamma p^2 3^{-p}]^{p+1}-1$$

$$\leqq \text{const}\,\frac{C\gamma p^5}{3^p} \leqq \tfrac{1}{2},$$

for all large p. Then $\frac{1}{2} \leqq |D/V| \leqq \frac{3}{2}$, so that $D \neq 0$. Similarly, $|D'/V'| \leqq \frac{3}{2}$. This yields

$$\left|\frac{D'}{D}\right| \leqq 2\left|\frac{D'}{V}\right| = 2\left|\frac{D'}{V'}\right| \left|\frac{V'}{V}\right| \leqq 3M = B,$$

as required.

Formula (3.3) shows that we can obtain the determinant C from the determinant D by multiplying its i-th row by the factor

$$(-1)^i \frac{1}{(2i+1)!} 2^{2i} 3^{(2i+1)p}, \quad i = 0, \cdots, p,$$

and by multiplying the k-th column of D by $(-1)^K$. Similarly with C' and D'. Hence

$$\left|\frac{C_{ik}}{C}\right| = \left|\frac{D_{ik}}{D}\right| \frac{(2i+1)!}{2^{2i}} 3^{-(2i+1)p}.$$

Here $i \leqq p$, and we obtain, using (3.4),

$$\left|\frac{C_{ik}}{C}\right| \leqq B\, 3^{-p}, \quad i = 0, \cdots, p; \quad k = p, \cdots, 2p. \tag{3.5}$$

We define inductively integers p_l so that the intervals $[p_l, 2p_l]$, $l = 1, 2, \cdots$ are disjoint, and numbers b_k satisfying $|b_k| \leqq k^{-2}$ for $p_l \leqq k \leqq 2p_l$, $l = 1, 2, \cdots$. Outside of the intervals we put $b_k = 0$. The b_k are selected in such a way that the coefficients of x^{2i+1}, $i = 0, \cdots, p_l$ in the sum $S_l(x) = \sum_{k \leq 2p_l} b_k T_{3^k}(x)$ are zero. Then S_l is a polynomial of form (1.1) with $s \geqq 2p_l + 3$, of degree $n = 3^{2p_l}$. We have $s \geqq (\log 3)^{-1} \log n$. An appeal to Theorem 1 would then complete the proof.

Let $p_1, \cdots, p_{l-1}$ and the corresponding b_k be already known. Let ρ_i denote the coefficient of x^{2i+1} in the polynomial S_{l-1}; it is zero for $i > 3^{2p_{l-1}}$ (and for $i \leqq p_{l-1}$). At step l, we select p_l so large that

$$\begin{cases} 2p_{l-1} < p_l \\ B\rho < p_l^{-2} 3^{p_l}, \quad \rho = \sum_i |\rho_i|. \end{cases} \tag{3.6}$$

The condition that the polynomial S_l does not contain x^{2i+1} for $i \leqq p_l$ leads to the system of equations

$$\rho_i + \sum_{p_l \leqq k \leqq 2p_l} c_{ik} b_k = 0, \quad i = 0, \cdots, p_l \tag{3.7}$$

for the b_k. This system is solvable, since its determinant $C \neq 0$. For the b_k we get by (3.5) and (3.6)

$$b_k = -\sum_{i=0}^{p_l} \frac{C_{ik}}{C} \rho_i,$$

$$|b_k| \leqq \rho B 3^{-p_l} < p_l^{-2} \leqq k^2,$$

proving Theorem 3.

Added in proof (November 10, 1977). Saff and Varga [3] have recently established that $\Delta(\theta) = \theta^2$.

References

1. G. G. Lorentz, *Approximation by incomplete polynomials, problems and results*, in Proceedings of Conference on Rational Approximation, Tampa, Florida, December 1976, Academic Press, to appear.

2. T. J. Rivlin, *The Chebyshev Polynomials*, John Wiley and Sons, New York, 1974.

3. E. B. Saff and R. S. Varga, *The sharpness of Lorentz's theorem on incomplete polynomials*, preprint.

Department of Mathematics
The University of Texas
Austin, Texas 78712 USA